Attack and Release Time Constants in RMS-Based Compressors and Limiters

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Although RMS detector time constants are often set by ear, an art in itself, the importance of a mathematical "tool" is explained. The mathematics of the RMS detector is revisited and models for time and frequency behavior are determined.

0 Introduction

The Root Mean Square, or RMS, value of an *ac* voltage or current is the equivalent *dc* voltage or current that generates the same amount of real power in a resistive load. In other words, RMS is the square root of the integral of the square of the *ac* signal, over a period of time and weighted by the same length of time. The mathematical formula follows:

$$V_{rms} = \lim_{T\to\infty}\sqrt{\frac{1}{T}\int_{-\infty}^{T} v_{in}^2(t)\cdot dt} \qquad (1)$$

where V_{rms} is the *RMS* value of the input waveform, $v_{in}(t)$, and T is the time at which the measurement is made.

A true RMS detector leads to a *dc* output in conformance with equation (1), independent of the input waveform. A quasi RMS detector can be built by scaling the output of a peak or average detector for a specific waveform, such as a cosine. In the case of the peak detector the scaling factor is the crest factor defined as the ratio of peak to RMS value. In particular, for a cosine, the crest factor is the square root of two.

RMS detectors have evolved over time using different technologies. One of the first technologies was to use the signal to be measured to heat a resistive element and measure the element's change in temperature. This method is still used in high frequency true RMS detectors. While this kind of detector has a wide frequency bandwidth, and such designs have been integrated on silicon [1], this method is not suitable for audio. The main disadvantages are limited dynamic range of 40 to 60 dB and a slow attack time constant, making it inadequate for many audio protection applications.

Advances in bipolar technology in the 1970's made possible the introduction of a solid state true RMS detector which does not use heating to make the measurement. Dynamic range of such detectors is better than 80dB, and an attack time of less than *1ms* is possible. Thus, this paper concentrates on silicon-based detectors which do not use thermal methods to measure RMS value.

1.0 The RMS detector

1.1 Building blocks

Definition (1) implies that the integration time is infinite. This is not practical for any real time application. Therefore, a tradeoff must be made. From filter theory, it is shown that the output of a low pass filter approximates the integral of its input signal. However, the integration time is not infinite and it is limited by the time constant of the filter. This concept is applied here.

Fig 1.a shows a block diagram of an RMS detector which provides a practical approximation to definition (1). The input voltage is squared and then applied to the first order low pass filter made of resistor R and capacitor C. The following equations can be deduced:

$$v_1(t) = v_{in}^2(t) \tag{2}$$

$$\frac{v_1(t) - V_2(t)}{R} = C\frac{dV_2(t)}{dt} \tag{3}$$

where it is assumed that the square and square root blocks have *infinite* input impedance and *zero* output impedance.

Substituting (2) into (3) and using notation $\tau = \frac{1}{RC}$, the following differential equation can be written:

$$v_{in}^2(t) = \tau\frac{dV_2(t)}{dt} + V_2(t) \tag{4}$$

In the next section it is shown that the solution $V_2(t)$ of the differential equation (4) is linearly proportional to the integral of the square of the input voltage and thus $V_{rms}(t)$ is proportional to the true RMS level of $v_{in}(t)$. However, for most audio applications, a logarithmic representation of the RMS output level is more useful. The logarithm function provides a natural compression of the *ac* input dynamic range, so that the output signal is proportional in decibels (dB). Using a minimal interface, an LED or analog display can be attached to the detector output, realizing an RMS meter. A compressor or expander can easily be achieved by connecting the detector output to a Voltage Controlled Amplifier (VCA) with an exponential control characteristic. VCAs with gain control directly in dB are readily available on the market.

Fig. 1.b shows the simplified circuit diagram of a log-responding level detector [2]. The input voltage is full wave rectified, converted to a current, and applied to *OA1*, *D1* and *D2*. Let us assume that *D1* and *D2* have the same characteristics, most importantly the same saturation current I_S. *D1* and *D2* in the negative feedback path of the operational amplifier *OA1* convert the input current, $i_{in}(t)$, to a logarithmic voltage:

$$v_1(t) = 2V_T \ln\left(\frac{|i_{in}(t)|}{I_S}\right) = V_T \ln\left(\frac{i_{in}^2(t)}{I_S^2}\right) \tag{5}$$

where I_S is the diode saturation current and V_T is the thermal voltage ($\frac{kT}{q} = 25.9mV$, at room temperature). Also, notice that the identity $|i_{in}(t)|^2 = i_{in}^2(t)$ is used.

Therefore, $v_1(t)$ is proportional to the logarithm of the square of the input current. The voltage $v_1(t)$ present at the OA1 output needs to be integrated. There are two ways to accomplish the integration. One way is to exponentiate the voltage $v_1(t)$ and filter it in the linear domain. The square root operation is accomplished by taking the logarithm of the output voltage and using it as a negative feedback to the logarithmic block. In this way, the output voltage is subtracted from the squared input voltage in the log domain. Due to the properties of logarithms, this scheme performs the square root operation. This technique is explained in more detail in reference [3].

Another way to integrate $v_1(t)$ is to a filter it in the *log domain*. In Fig. 1.b, $D3$, C_T and I_T serve as a first order log domain low pass filter. The description of log domain filters is elaborate and is beyond the scope of this paper. Log domain filters are covered in reference [4] in more detail. However, for this circuit let us identify the currents through each component of the log filter.

The current in D3 is exponential with the voltage across it:

$$I_{D3} = I_S \exp\left(\frac{v_1(t) - V_2(t)}{V_T}\right) \tag{6}$$

The current in the capacitor C_T is proportional to the instantaneous change of voltage across it:

$$I_{CT} = C_T \frac{dV_2(t)}{dt} \tag{7}$$

So, using Kirchoff's Current Law (KCL), the current in $D3$ is:

$$I_{D3} = I_S \exp\left(\frac{v_1(t) - V_2(t)}{V_T}\right) = C_T \frac{dV_2(t)}{dt} + I_T \tag{8}$$

where C_T is the timing capacitor and I_T is an external current source that sets the timing current[1].

Substituting for $v_1(t)$ in equation (8) using equation (5) yields:

$$i_{in}^2(t) = I_S C_T \exp\left(\frac{V_2(t)}{V_T}\right) \frac{dV_2(t)}{dt} + I_S I_T \exp\left(\frac{V_2(t)}{V_T}\right) \tag{9}$$

The last block in Fig. 1, $OA2$, $D4$ and current source I_B, is just an output buffer and a level shifter. Thus the *dc* output voltage is:

$$V_{rms}(t) = V_2(t) - V_T \ln\left(\frac{I_B}{I_S}\right) \tag{10}$$

Ideally the long term RMS output is a pure *dc* voltage. As shown in Fig 1.b, $V_2(t)$ is the result of the integration of the log filter. However, $V_2(t)$ is still a function of time because it carries

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1 It is interesting to note that on a transient basis, the current through $D3$ and C_T is linearly proportional to the square of the input current.

transient and ripple information due to the finite integration time. Consequently, $V_{rms}(t)$ in equation (10) varies with time, also. Equation (10) can be solved for $V_2(t)$. Then, substituting for $V_2(t)$ in equation (9), and rearranging terms the following equation is obtained:

$$\frac{i_{in}^2(t)}{I_R^2} = \tau \frac{d}{dt}\left(\exp\left(\frac{V_{rms}(t)}{V_T}\right)\right) + \exp\left(\frac{V_{rms}(t)}{V_T}\right) \qquad (11)$$

where the following notations were used:

$$I_R^2 = I_B I_T \qquad (12); \qquad \tau = \frac{C_T V_T}{I_T} \qquad (13)$$

where I_R is the reference level and τ is the integrator time constant.

The reference level, I_R, will be the input current for which output voltage V_{rms} is *zero* volts.

Let us define the following function:

$$y(t) = \exp\left(\frac{V_{rms}(t)}{V_T}\right) \qquad (14)$$

Substituting (14) into (11), the differential equation can now be written as:

$$\frac{i_{in}^2(t)}{I_R^2} = \tau \frac{dy(t)}{dt} + y(t) \qquad (15)$$

Notice the similarity between equations (4) and (15). Both have the familiar form of the first order low pass filter differential equation [4]. Appendix I describes how to solve this first order differential equation.

From definition (14) it can be seen that $V_{rms}(t)$ is proportional to the logarithm of $y(t)$. In the next section it is shown that $V_{rms}(t)$ is proportional to the logarithm of the square of the RMS level of the input current. Since the RMS output level is in a logarithmic format, the square root operation can be easily done by dividing $V_{rms}(t)$ by two. In Fig. 1.b two equal resistors perform this operation. As a result, V_{rms1} is proportional to the logarithm of the RMS level of the input current.

1.2 General solution

Equations (4) and (15) can be solved using the same method. Appendix I describes in detail how to solve both equations. The general solution of differential equation (4) follows:

$$V_2(t) = \left(\int \frac{1}{\tau} v_{in}^2(t) \cdot \exp\left(\frac{t}{\tau}\right) \cdot dt + c\right) \cdot \exp\left(-\frac{t}{\tau}\right) \qquad (16)$$

where c is a constant.

The general solution of equation (11) follows:

$$\exp\left(\frac{V_{rms}(t)}{V_T}\right) = \left(\int \frac{1}{\tau} \frac{i_{in}^2(t)}{I_R^2} \exp\left(\frac{t}{\tau}\right) \cdot dt + c\right) \exp\left(-\frac{t}{\tau}\right) \qquad (17)$$

where c is a constant.

The solutions of the RMS detectors shown in Fig. 1.a and Fig. 1.b show a strong resemblance. The main difference is that in (17) the solution is in an exponential format. Thus, both circuits perform the same function of integrating the square of the input voltage and current, respectively.

Furthermore, if the logarithm function is applied to both sides of (17) the following result is found:

$$V_{rms}(t) = -\frac{V_T}{\tau}t + 2V_T \ln\left(\sqrt{\frac{1}{\tau}\int \frac{i_{in}^2(t)}{I_R^2}\exp\left(\frac{t}{\tau}\right)dt + c}\right) \tag{18}$$

Also, note that equation (17) satisfies the definition of the reference level. Indeed, if the input current RMS level equals the reference level, $RMS(|i_{in}(t)|)=I_R$, then $V_{rms}(t\rightarrow\infty)=0$, for $c=0$.

Equation (18) has some limitations in explaining the real detector since it describes an ideal RMS detector. That is where constant c helps. It has different forms for different initial conditions, e.g., attack or release. Particular values of constant c are discussed in chapter 1.4 .

1.3 Particular solution

The general solution has particular solutions for specific input waveforms. One interesting case is when the input voltage has a sinusoidal function. Therefore let $v_{in}(t)$, in Fig. 1.a and 1.b , be defined as:

$$v_{in}(t) = V_0\cos(\omega t) \tag{19}$$

where V_0 is the peak voltage and ω is the frequency.

The input voltage has no *dc* component added because the RMS detector input is usually *ac* coupled [2]. The input current, $i_{in}(t)$, for the logarithmic RMS detector in Fig 1.b, is the ratio of the input voltage to the input resistor.

$$i_{in}(t) = \frac{v_{in}(t)}{R_{in}} = \frac{V_0\cos(\omega t)}{R_{in}} = I_0\cos(\omega t) \tag{20}$$

where I_0 is defined as the ratio of the peak input voltage to the input resitor.

Let us substitute $v_{in}(t)$ from equation (19) into equation (16). The solution for the linear detector, assuming $t >> \tau$, is:

$$V_2(t) = \frac{V_0^2}{2}G(\omega, t) \tag{21}$$

Let us substitute $i_{in}(t)$ from equation (20) into equation (18). The solution for the logarithmic detector, assuming $t >> \tau$, is:

$$V_{rms}(t) = V_T \ln\left(\left(\frac{I_0}{\sqrt{2}I_R}\right)^2 G(\omega, t)\right) = V_T \cdot \ln\left(\left(\frac{I_0}{\sqrt{2}I_R}\right)^2\right) + V_T \cdot \ln(G(\omega, t)) \tag{22}$$

where $G(\omega,t)$ is the following expression (see Appendix I):

$$G(\omega,t) = 1 + \frac{\cos(2\omega t) + 2\omega\tau\sin(2\omega t)}{1 + 4\omega^2\tau^2} \tag{23}$$

The real RMS level is calculated by taking the square root. In the case of the circuit in Fig. 1.a, this is accomplished by the last block. Thus the RMS output voltage, V_{rms}, is:

$$V_{rms}(t) = \frac{V_0}{\sqrt{2}}\sqrt{G(\omega,t)} \tag{24}$$

In the case of the logarithmic detector in Fig. 1.b, a simple division by two performs this operation. Again, recalling that the logarithm of a product of terms is equal to the sum of the logarithms of each term, the RMS detector output voltage, V_{rms1}, is:

$$V_{rms1}(t) = \frac{V_T}{2}\ln\left(\left(\frac{I_0}{\sqrt{2}I_R}\right)^2 G(\omega,t)\right) = V_T\ln\left(\frac{I_0}{\sqrt{2}I_R}\right) + V_T\ln\left(\sqrt{G(\omega,t)}\right) \tag{25}$$

Equations (24) and (25) show that the output voltage of both detectors is proportional to the RMS level of the input signal. Obviously, the output voltage of the detector in Fig. 1.a is linearly proportional to the RMS level of the input signal. On the other hand, the output voltage of the detector in Fig. 1.b is proportional to the logarithm of the RMS level of the input signal.

Because, an RMS detector with output voltage proportional to the logarithm of the RMS level is more useful in compressor-limiter applications, the next sections focus only on log-responding RMS detectors.

Furthermore, the final square root operation requires only a change in scaling the RMS (log) signal by a factor of half. Therefore, only V_{rms} as shown in Fig. 1.b and given by equation (18), is considered for further discussion.

Equation (22) does not have any transient information since $t >> \tau$. It can be noted that the particular solution has two terms: a *dc* term and an *ac* term or ripple.

1.3.1 Particular solution *dc* term

The output *dc* part, as expected for a sinusoidal input signal, is proportional to the logarithm of the peak level divided by the square root of two. Ideally, only the *dc* term is wanted. The ripple is an error factor that can be minimized by increasing the time constant. From equation (22) scale and temperature constants can be extracted. If ripple is neglected, then equation (22) can be rewritten as:

$$V_{rms} = \frac{V_T\ln(10)}{10}\left(20\log\left(\frac{I_0}{\sqrt{2}I_R}\right)\right) \overset{Temp=300K}{=} 5.96\cdot\left(\left[RMS(i_{in}(t))\right]_{dB} - \left[I_R\right]_{dB}\right)\left[\frac{mV}{dB}\right] \tag{26}$$

Therefore, the relationship between input current, in dB, and RMS output voltage, in *mV*, is linear with a theoretical slope of ***5.96 mV/dB*** at room temperature. In reality, commercial detector chips run at a higher internal temperature and the slope can vary from ***6.1 mV/dB*** to ***6.5 mV/dB*** at room temperature, depending on the power dissipation of the detector used [2] [6]. In compressor or expander applications where the RMS detector is connected to a VCA with matching *mV* to *dB* type linear conversion, it is very important that both devices have the same slope. Otherwise said, both should have the same internal temperature in order to track properly. Therefore a good compressor or expander design should use an RMS detector and a VCA from the same family. The best scenario is when both devices are integrated on the same substrate and have guaranteed temperature tracking [6] [7].

In other applications where circuitry is included between the RMS detector and VCA, e.g., gate, limiter threshold, or there is no VCA, e.g., metering, external temperature compensation should be provided. The temperature coefficient can be calculated from thermal voltage $V_T = \frac{kT}{q}$ as being *+3300 ppm/℃* referred to room temperature. In Fig. 2, two ways of compensating for the RMS detector output drift with temperature are presented. The temperature correction element is a resistor with a positive temperature coefficient of *+3300 ppm/℃*. Unfortunately the temperature-compensated resistors are commonly available only in 5% tolerance, so scale calibration is usually required. Fig. 2.a presents a common circuit for temperature compensation [3]. While it is simple to implement, it has a major disadvantage that is the potentiometer temperature coefficient can affect the temperature tracking of the special resistor. A solution to this problem is the circuit in Fig. 2.b where the dependence to the potentiometer temperature coefficient is significantly reduced.

1.3.2 Particular solution *ac* term

The second term of the last part of equation (22) is the *ac* component of the RMS detector output voltage, $v_{rms}(t)$. An interesting characteristic of the log domain filter is that the amplitude of the ripple is independent of input level. In other words, for the same frequency, $1V_{rms}$ and $1mV_{rms}$ input signals generate the same amount of peak-to-peak ripple at the output of the RMS detector. This behavior can be explained by the property of the logarithmic function that splits the product into a sum (see Appendix I, equation (AI 14)). Therefore the *ac* term does not multiply the *dc* part.

The *ac* component contains a combination of second harmonic sine and cosine of the input signal. For a better understanding of the *ac* spectrum at the detector output, an approximation of the logarithmic function is done. A logarithm can be expanded in series as follows:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... = \sum_n (-1)^{n+1} \frac{x^n}{n} \qquad (27)$$

where $n=1,2,3,...$.

For small x the sum can be approximated to the first term. In this case x is the second summation term of equation (23). For $\omega\tau >> 1$, equation (23) can be approximated to $G(\omega,t) = 1 + \frac{sin(2\omega\, t)}{2\omega\tau}$. In this case $x = \frac{sin(2\omega\, t)}{2\omega\tau}$. If $x < 0.1$ then logarithmic expansion (27) can be applied to the second term of the last part of equation (22), $V_T ln(G(\omega,t))$. A commonly used time constant is $\tau = 35ms$ [2] [6]. For this particular time constant, $V_T ln(G(\omega,t))$ can be approximated to the first logarithmic expansion term for frequencies greater than *22 Hz*. Thus, using the definition of time constant in equation (13), the expression for ripple becomes:

$$v_{ripple}(\omega, t) \approx V_T \frac{\sin(2\omega t)}{2\omega\tau} = \frac{I_T}{2C_T}\frac{1}{\omega}\sin(2\omega t) \tag{28}$$

It is interesting to note that although the input signal was full wave rectified, the ripple is a pure second harmonic sinusoid. From equation (28) it can be seen that the ripple decreases with ***ω*** at a rate of ***-6 dB/octave***. The amount of RMS ripple measured at the RMS detector output is [8]:

$$V_{ripple} = \frac{I_T}{\sqrt{2}C_T}\frac{1}{\omega} \tag{29}$$

All the above approximations are useful for audio applications where the time constant is in the range of tens of milliseconds. However, in some applications, independent control of attack and release time constants is required. Most of these applications use the RMS detector with very short time constants, and provide further processing to shape the transient responses outside the detector. For these cases, the logarithm expansion cannot be approximated to the first series term.

At frequencies lower than $\frac{1}{2\pi \cdot \tau}$ the combination of higher order harmonics produces a ripple that looks more like a full wave rectified cosine, although the shape is actually a combination of cosine, sine and logarithmic functions. However, at frequencies much higher than $\frac{1}{2\pi \cdot \tau}$ equation (28) is still valid and the *ac* component is a second harmonic sinusoid. Fig. 3 shows the difference of the ripple shape at low and high frequency. The time constants used, *17μs*, is four orders of magnitude smaller than commonly used for RMS detectors. The *Y* scale is altered in order to magnify the high frequency ripple.

Any external control of the time constant implements a low pass filter topology that averages the output voltage V_{rms}. The average of a *dc* voltage is the same voltage. Consequently, in long term, the first *dc* element of equation (22) is not affected by an external averaging process. The effect of an outside lowpass on the *ac* component is a different matter.

Assuming an external linear low pass filter as the averaging circuit, and sinusoidal ripple, then the average, over a period of time longer than the signal period, is *zero*. This result is very intuitive since there is an equal number of identical peaks and troughs, over a long period of time, around a *dc* offset (see equations (22) and (28)). Therefore, for long time constants ripple remains a second harmonic sinusoid, and the average *dc* output of the RMS detector is not affected by the ripple over the audio band. In contrast, a fast time constant changes the shape of the ripple and the peaks and valleys are not symmetric anymore (see Fig. 3). The averaging process will then add a small *dc* component in proportion to the asymmetry of the ripple. This phenomenon is called *dc error*. The *dc error* is larger at low frequencies where the asymmetry is magnified.

1.3.3 The *dc* error

The *dc error* can be calculated from the ripple expression:

$$v_{ripple}(\omega, t) = V_T \ln\left(1 + \frac{\cos(2\omega t) + 2\omega\tau \sin(2\omega t)}{1 + 4\omega^2\tau^2}\right) \quad (30)$$

Let us define φ as follows:

$$\tan(\varphi) = 2\omega\tau \quad (31)$$

Then, equation (30) can be written as:

$$v_{ripple}(\omega, t) = V_T \ln(1 + \cos(\varphi)\cos(2\omega t - \varphi)) \quad (32)$$

and φ can be calculated from equation (31):

$$\varphi = \arctan(2\omega\tau) \quad (33)$$

Equation (32) can be expanded in series in the same manner as equation (27):

$$v_{ripple}(\omega, t) = V_T \sum_n (-1)^{n+1} \frac{\cos^n(\varphi)\cos^n(2\omega t - \varphi)}{n} \quad (34)$$

where $n=1,2,3,\ldots$.

This last equation indicates the harmonic content at the RMS output. For a large time constant $\tan(\varphi) = 2\omega\tau \to \infty$, therefore $\varphi \to (\pi/2)$ and $\cos\varphi \to 0$. In this case only the first term of the sum $n = 1$ is significant. For low frequencies and small time constants, $\cos\varphi$ becomes significant and more and more terms should be taken into consideration.

As discussed earlier, if an external low pass filter is used, it integrates equation (34). Let us assume that the external low pass filter has a time constant τ_2. The *dc error* is the average of equation (34):

$$er_{dc} = \frac{V_T}{\tau_2} \sum_n (-1)^{n+1} \frac{\cos^n(\varphi)}{n} \int_0^{\tau_2} \cos^n(2\omega t - \varphi) \cdot dt \quad (35)$$

Equation (35) can be rearranged as a sum of even and odd terms as follows:

$$er_{dc} = \frac{V_T}{\tau_2} \sum_n \left(\frac{\cos^{2n-1}(\varphi)}{2n-1} \int_0^{\tau_2} \cos^{2n-1}(2\omega t - \varphi) \cdot dt - \frac{\cos^{2n}(\varphi)}{2n} \int_0^{\tau_2} \cos^{2n}(2\omega t - \varphi) \cdot dt \right) \quad (36)$$

Equation (36) is difficult to evaluate. The sum and integration create nested sums that take a long time to compute. Fortunately equation (36) can be simplified. Let us consider the following equivalent integral:

$$E = \int \cos^n(y) \cdot dy \quad (37)$$

Using partial integration, the following reduction formula is obtained (see Appendix II):

$$\int \cos^n(y) \cdot dy = \frac{\sin(y)}{n} \cos^{n-1}(y) + \frac{n-1}{n} \int \cos^{n-2}(y) \cdot dy \quad (38)$$

In Appendix II, it is demonstrated that for n odd equation (38) becomes:

$$E_{odd} = \sin(y)\left(c_1 \cos^{n-1}(y) + c_2 \cos^{n-3}(y) + c_3 \cos^{n-5}(y) + \cdots + c_{\frac{n+1}{2}}\right) \qquad (39)$$

where c_i are real coefficients. For the exact values see Appendix II.

In Appendix II it is demonstrated that equation (39) integrated over one signal period is *zero*. The intuitive explanation is that function *sin(y)* has equal peaks and troughs over a period of time and multiplies a sum of higher harmonics. Therefore, there is no contribution from odd terms in equation (36). For *n* even equation (38) becomes:

$$E_{even} = \sin(y)\left(c_1 \cos^{n-1}(y) + c_2 \cos^{n-3}(y) + c_3 \cos^{n-5}(y) + \cdots + c_{\frac{n}{2}} \cos(y)\right) + c_{\frac{n}{2}+1}\, y \qquad (40)$$

In this case there is an extra *dc* term at the end of equation (40). Integrated over the one signal period everything that is multiplied by *sin(y)* is *zero*. The only not zero expression is the last coefficient, $c_{\frac{n}{2}+1}$. The formula for the last coefficient was deduced in Appendix II as follows:

$$c_{\frac{n}{2}+1} = \frac{(n-1)(n-3)(n-5)\cdots 3 \cdot 1}{n(n-2)(n-4)\cdots 4 \cdot 2} \qquad (41)$$

Taken into account all results from equations (38), (39), (40) and (41), equation (36) can be simplified as:

$$er_{dc} = -V_T \sum_n \frac{\cos^{2n}(\varphi)}{2n} \frac{(2n-1)(2n-3)(2n-5)\cdots 3 \cdot 1}{2n(2n-2)(2n-4)\cdots 4 \cdot 2} \qquad (42)$$

where $n=1,2,3,\ldots$.

Since $\cos(\varphi)$ is raised to an even power it is positive for any value of φ. Note that the *dc error* is always negative. Additionally, the *dc error* is proportional to the thermal voltage V_T which means that the *dc error* increases with temperature at a rate of ***+3300 ppm/°C***[2].

$Cos(\varphi)$ can be written as a function of $tan(\varphi)$ and replacing $tan(\varphi)$ by its definition (31):

$$\cos^2(\varphi) = \frac{1}{1 + tg^2(\varphi)} = \frac{1}{1 + 4\omega^2\tau^2} \qquad (43)$$

and the *dc error* becomes an explicit function of frequency ω and time constant τ:

$$er_{dc} = -V_T \sum_n \frac{1}{2n\left(1 + 4\omega^2\tau^2\right)^{2n}} \frac{(2n-1)(2n-3)(2n-5)\cdots 3 \cdot 1}{2n(2n-2)(2n-4)\cdots 4 \cdot 2} \qquad (44)$$

Fig. 4 shows a plot of equation (44) for a sum of different number of terms, 5, 20 and 100 using a fast time constant of $\tau = 17\mu s$. Unfortunately, the precision of equation (44) increases slowly with the number of iterations. As can be seen in Fig. 4, the precision is better than 90%

[2] In general the ripple has the same temperature dependence as the RMS output.

beyond 20 iterations. As a practical matter, to compute 20 or more iterations requires the use of a computer.

It is possible to use equation (44) to predict *dc error* vs. frequency for different time constants. Fig. 5 shows *dc error* vs. frequency for two different time constants. In both cases 20 iterations were used. For the common time constant of *35ms*, the *dc error* is almost *zero* for the full audio bandwidth. For the fast time constant of *17μs*, the error approaches *-15mV* (*-2.5dB*) at low frequencies.

1.4 Transient response

So far, the transient response at the RMS detector output has been ignored. The transient solution of equation (18) for the input signal given by (20) is:

$$V_{rms}(t) = V_T \ln\left(\frac{I_0^2}{2I_R^2} G(\omega, t) + c \cdot \exp\left(-\frac{t}{\tau}\right)\right) \tag{45}$$

where $G(\omega,t)$ is the *ac* component at the detector output given by (23).

As described in the previous sections, $G(\omega,t)$ adds ripple to the *dc* output which for large time constants is small. Constant c is found by setting boundary conditions at $t = 0$ and $t \to \infty$. With no signal at the input, a real RMS detector does not have $-\infty$ output voltage, as equation (18) might suggest. In commercial devices, the typical *dc* voltage at the detector output in the absence of input signal is $\approx$*-400mV*. This plateau is determined by the RMS level of the detector's input noise, caused by internal noise sources, input resistor noise and detector input bias currents. The input noise and input bias currents are not under the designer's control. However, the external input resistor is. Higher input resistor values means more noise at the detector input and a higher plateau voltage. Since the maximum input voltage is limited by the application voltage rails, raising the plateau voltage reduces the detector dynamic range. At any rate, let us define the plateau voltage as V_P.

1.4.1 Attack

The following boundary condition is true:

$$V_{rms} = V_P, \ t = 0 \tag{46}$$

Substituting condition (46) in equation (45), c has the following form:

$$c = \exp\left(\frac{V_P}{V_T}\right) - \frac{I_0^2}{2I_R^2} G(\omega,0) \tag{47}$$

By replacing equation (47) in equation (45) V_{rms} has the following solution:

$$V_{rms}(t) = V_T \ln\left[\frac{I_0^2}{2I_R^2}\left(G(\omega,t) - G(\omega,0)\exp\left(-\frac{t}{\tau}\right)\right) + \exp\left(-\frac{t}{\tau}\right)\exp\left(\frac{V_P}{V_T}\right)\right] \tag{48}$$

Relation (48) is the general solution of equation (18) for attack and particular input signal defined in (20).

In equation (23) it can be noted that the second term of $G(\omega,t)$ is inverse proportional to frequency ω. For a commonly used time constant , $\tau = 35ms$, the maximum value of $G(\omega,t)$ at $20Hz$ is close to *one*. In order to make the following equations more clear and to better understand the time dependence of the RMS detector output voltage, it can be assumed that $G(\omega,t) = 1$. Therefore, equation (48) can be written as:

$$V_{rms}(t) = V_T \ln\left[\frac{I_0^2}{2I_R^2}\left(1-\exp\left(-\frac{t}{\tau}\right)\left(1-\frac{2I_R^2}{I_0^2}\exp\left(\frac{V_P}{V_T}\right)\right)\right)\right] \qquad (49)$$

Equation (49) does not explicitly show the time dependence at the detector output. Therefore the following approximations can be made:

For $t << \tau$:

$$\exp\left(-\frac{t}{\tau}\right) \cong 1-\frac{t}{\tau} \qquad (50)$$

Combining (50) and (49) :

$$V_{rms}(t) = V_P + V_T \ln\left(1+\frac{t}{\tau}\left(\frac{I_0^2}{2I_R^2}\exp\left(-\frac{V_P}{V_T}\right)-1\right)\right) \sim \ln(1+\frac{t}{\tau}) \qquad (51)$$

Therefore the start of the attack is proportional to $ln(1+\frac{t}{\tau})$. Also it is directly proportional to the logarithm of the square of the input peak current and inversely proportional to the logarithm of the time constant τ.

For $t >> \tau$, $exp(-\frac{t}{\tau})$ is very small. So series approximation (27) can be applied to (49). Combining (27) with (49):

$$V_{rms}(t) = V_T\left(\ln\left(\frac{I_0^2}{2I_R^2}\right)-\exp\left(-\frac{t}{\tau}\right)\left(1-\frac{2I_R^2}{I_0^2}\exp\left(\frac{V_P}{V_T}\right)\right)\right) \sim 1-\exp\left(-\frac{t}{\tau}\right) \qquad (52)$$

Equation (52) shows that the end of the attack is proportional to $1 - exp(-\frac{t}{\tau})$.

In Fig. 6.a, equations (49) (solid line) and (51) (dashed line) are plotted for time constant $\tau = 35ms$, plateau voltage $V_P = -400mV$, reference level $I_R = \frac{100mV}{R_{in}}$, and input level ten times, *+20dB*, above the reference level. In long term the RMS output voltage is $\approx 120mV$. The dashed line represents the initial fit for $t << \tau$. Fig. 6.b shows equations (49) (solid line) and (52) (dashed line). Equation (52) is the approximation for $t >> \tau$. Fig. 6.c and Fig. 6.d show the influence of the ripple for two different time constants, *35 ms* and $17\mu s$. The solid line represents the approximation of the general solution for $G(\omega,t)=1$, equation (49), and the dashed line represents the general solution, equation (48).

1.4.2 Release

When the detector input signal is removed (or decreased), the RMS output does not recover right away, because the timing capacitor does not discharge instantaneously. Therefore, the RMS output voltage can be determined by solving equation (17) for a new set of initial conditions. Let us assume that the input current, $i_{in}(t)$ is *zero* for $t \geq 0$. Then, the integral in equation (17) is *zero* as well. Equation (17) can be written as:

$$\exp\left(\frac{V_{rms}(t)}{V_T}\right) = c \cdot \exp\left(-\frac{t}{\tau}\right) \tag{53}$$

Note that for $t \rightarrow \infty$ the detector output voltage $V_{rms} \rightarrow -\infty$. A practical RMS detector does not have $-\infty$ output voltage. As explained in section 1.4, the RMS detector output voltage will eventually reach a plateau voltage, V_P. In order to satisfy this condition a new constant should be added to equation (53).

$$\exp\left(\frac{V_{rms}(t)}{V_T}\right) = c \cdot \exp\left(-\frac{t}{\tau}\right) + c' \tag{53a}$$

Also, let us assume that for $t < 0$ the RMS detector output voltage is given by the attack time equation (49), for a very long time. Therefore, for $t < 0$ the detector output voltage is $V_T ln\left(\frac{I_0^2}{2I_R^2}\right)$.

The new set of conditions is:

For $t = 0$:

$$\frac{I_0^2}{2I_R^2} = c + c' \tag{54}$$

For $t \rightarrow \infty$:

$$\exp\left(\frac{V_P}{V_T}\right) = c' \tag{55}$$

The system of equations (54) and (55) can be solved for c and c'. Substituting the two constants into (53a), $V_{rms}(t)$ during release time has the following relation:

$$V_{rms}(t) = V_T \ln\left[\exp\left(\frac{V_P}{V_T}\right) + \left(\frac{I_0^2}{2I_R^2} - \exp\left(\frac{V_P}{V_T}\right)\right)\exp\left(-\frac{t}{\tau}\right)\right] \tag{56}$$

In order to understand the time dependence of relation (56) it can be safely assumed that $exp(\frac{V_P}{V_T})$ is very small. Indeed, for $V_P = -400mV$, $exp(\frac{V_P}{V_T}) = 1.96 \cdot 10^{-7}$. However, the second summation term of the logarithm in equation (56) decreases exponentially with time. Therefore,

$exp(\frac{V_P}{V_T})$ will be significant after some period of time. This specific period of time during which $exp(\frac{V_P}{V_T})$ can be neglected, can be determined by imposing the condition that $exp(\frac{V_P}{V_T})$ is ten times smaller than the second summation term of the logarithm in equation (56). Thus, for $t \le \tau\left(ln\left(\frac{I_0^2}{20I_R^2} - exp\left(\frac{V_P}{V_T}\right)\right) - \frac{V_P}{V_T}\right)$, $exp(\frac{V_P}{V_T})$ can be ignored in equation (56). For $V_P = -400mV$ and input current $20dB$ above the reference level, $t \le 17.74\,\tau$. In this case $V_{rms}(t)$ is proportional to $-\frac{t}{\tau}$:

$$V_{rms}(t) = V_T \ln\left(\frac{I_0^2}{2I_R^2}\right) - V_T\frac{t}{\tau} \tag{57}$$

For $t > \tau\left(ln\left(\frac{I_0^2}{20I_R^2} - exp\left(\frac{V_P}{V_T}\right)\right) - \frac{V_P}{V_T}\right)$ the contribution of $exp(\frac{V_P}{V_T})$ can not be neglected anymore. In this case $exp(-\frac{t}{\tau})$ is very small and expansion in series (27) can be used. Consequently equation (56) is approximated as:

$$V_{rms}(t) = V_P + V_T \exp\left(-\frac{t}{\tau}\right)\left(\frac{I_0^2}{2I_R^2}\exp\left(-\frac{V_P}{V_T}\right) - 1\right) \tag{58}$$

Equations (56), (57) and (58) are plotted in Fig. 7 using a time constant $\tau = 35ms$, plateau voltage $V_P = -400mV$, reference level $I_R = \frac{100mV}{R_{in}}$ and input level ten times, $+20dB$, above the reference level for $t \le 0$. As it is expected the slope is quite linear until V_{rms} approaches V_P. The dashed line shows the linear approximation in (57) and the dash dot line shows the exponential approximation in equation (58).

In both cases, attack and release, it is assumed that the input level suddenly occurs and suddenly disappears, respectively, at the input. This is not the case with music material that has a large dynamic range. If input signal changes from one level I_0 to another I_1, in all the above formulae the plateau voltage V_P is replaced by $V_T ln\left(\frac{I_1^2}{2I_R^2}\right)$ and all results and plots remain valid.

2.0 Feedforward compressor

The major topologies for compressors and expanders are feedforward and feedback configurations [9]. Fig. 8 shows basic configuration of these processors. The main difference between feedforward and feedback is in which signal is presented to the detector input. In the feedback scheme, the detector "sees" the output level. One advantage of this topology is that in the case of a compressor, the output dynamic range is squeezed by the compression ratio, so the detector "sees" a smaller dynamic range. However, in feedback compressors *infinite* compression is theoretically impossible[3] [9].

The feedforward scheme overcomes this problem of infinite compression [9]. In fact even negative compression is possible with feedforward designs. Although the detector requires larger dynamic range at its input, high performance compressors can be built.

In Fig. 8, block k is just a *dc* gain for adjusting the compression ratio. The VCA transfer function is [5]:

$$g = \exp\left(-\frac{V_C}{2V_T}\right) \tag{59}$$

where V_C is the VCA gain control voltage, as shown in Fig. 8 . Also note that the VCA is a current in - current out device [5].

In the previous sections it was shown that besides the *dc* level the RMS detector output has two other important components. One is the long term *ac* ripple, superimposed on the *dc*; the other component is the transient, or short term behavior. In the following chapters the influence of long and short term components is analyzed.

2.1 Influence of RMS ripple

The VCA control voltage is:

$$V_C(t) = k \cdot V_{rms}(t) \tag{60}$$

where k is a constant.

Substituting (22), (23) and (60) in (59), assuming that $\omega\tau >> 1$, the VCA transfer function, or gain, is:

$$g(\omega, t) = \left(\frac{I_0^2}{2I_R^2}\left(1 + \frac{\sin(2\omega t)}{2\omega\tau}\right)\right)^{-\frac{k}{2}} \cong \left(\frac{I_0}{\sqrt{2}I_R}\right)^{-k}\left(1 - \frac{k}{2}\frac{\sin(2\omega t)}{2\omega\tau}\right) \tag{61}$$

It has been assumed that the compressor input is a cosine, then combining (20) with (61), the VCA output current is:

$$i_{out}(\omega t) = \left(\sqrt{2}I_R\right)^k I_0^{1-k}\left(\cos(\omega t + \alpha) - \frac{k}{8\omega\tau}\sin(3\omega t)\right) \tag{62}$$

where α is defined as:

$$\alpha = \arctan\left(\frac{k}{8\omega\tau}\right) \tag{63}$$

Theoretically, the compressor output contains only a third harmonic component in addition to the fundamental. The amplitude of the third harmonic is:

[3] Compression ratios of *10* or more approximate quite well an *infinite* compressor.

$$|I_{3\omega t}| = \frac{k}{8\tau}\frac{1}{\omega} \tag{64}$$

Therefore, the feedforward compressor distortion decreases at a rate of ***-6dB/octave*** . From formula (64), with infinite compression ($k = 1$) and $\tau = 35ms$, the distortion at *1kHz* is *0.057% (or -65dB)* and at *60Hz* is *0.95% (or -40dB)*. At lower frequencies (*20 - 50 Hz*) the distortion is underestimated by formula (64) because approximation (61) is no longer valid and more terms should be taken into account. In reality the distortion is slightly higher than predicted by formula (64). Also note that the distortion depends linearly on the compression ratio.

Of course, this evaluation assumes no distortions in the VCA and a perfect full-wave rectifier in the RMS detector.

2.2 Influence of RMS transient response

2.2.1 Attack

The transient response influences the envelope of the VCA output current. Since the RMS detector cannot respond instantaneously, the VCA gain will not change immediately when the signal is applied. Depending on the time constant used, for a very short time it is possible to see full waveform amplitude at the compressor output. Fortunately the initial rate of amplitude decrease is very high. Within *τ seconds*, the VCA gain will decrease significantly. Combining (49) and (60) in (59) and assuming (for simplicity) that the compressor threshold is set at the RMS reference level, implying $V_P = 0V$, the VCA gain during attack is described by the following equation:

$$g(t) = \left(\frac{I_0}{\sqrt{2}I_R}\right)^{-k}\left(1-\left(1-\frac{2I_R^2}{I_0^2}\right)\exp\left(-\frac{t}{\tau}\right)\right)^{-\frac{k}{2}} \tag{65}$$

For $t \to 0$, equation (65) can be approximated by:

$$g(t) \cong 1 - \frac{k}{2}\frac{t}{\tau}\left(\frac{I_0^2}{2I_R^2} - 1\right) \tag{66}$$

Therefore the initial rate of gain decrease (around $t = 0$) is proportional to the square of the peak of the input current and inversely proportional to the time constant τ. At the other extreme, for $t >> \tau$, the signal envelope decreases slowly at a rate given by the following formula:

$$g(t) \cong \left(\frac{I_0}{\sqrt{2}I_R}\right)^{-k}\left(1+\frac{k}{2}\left(1-\frac{2I_R^2}{I_0^2}\right)\exp\left(-\frac{t}{\tau}\right)\right) \tag{67}$$

The VCA output envelope during attack is pictured in Fig. 9.a , equation (65), with its two approximations equations (66) and (67). The final gain reduction is *-20dB*. Fig. 9.b shows a plot of equation (65) in decibels.

Often, an important compressor/limiter function is to protect a speaker driver. To determine whatever protection will be adequate, it is important to know the power delivered to the load during the attack time.

2.2.2 Release

After a large signal has been present, if the input decreases, the RMS detector does not react immediately. The capacitor of the log domain filter has to discharge. Therefore, during any sharp decrease, a compressor's output is squashed and slowly recovers. The relation for RMS output during release was demonstrated in section 1.4.2 . Combining equations (56) and (60) and substituting them in (59), the following result represents the VCA gain during release:

$$g(t) = \left(1 + \left(\frac{I_0^2}{2I_R^2} - 1\right)\exp\left(-\frac{t}{\tau}\right)\right)^{-\frac{k}{2}} \qquad (68)$$

Fig. 10.a shows the release envelope, equation (65), for an infinite compressor ($k = 1$), time constant *35ms* and *20dB* initial gain reduction. Fig. 10.b shows a plot of equation (65) expressed in decibels. The release shape and duration have a definite sonic impact. A fast release time produces a "pumping" amplitude modulation, sometimes along with noise modulation. Longer release time makes the signal gain changes harder to notice, but causes significant gain reduction for a long period of time (possibly seconds). It is more critical to set release time by ear than the attack time.

2.3 Influence of *dc* error

The *dc error* was calculated in section 1.3.3. As it can be seen in Fig. 5, the error at the RMS output at low frequencies is around *-15mV* for the case of a very fast time constant. Since the detector constant is ~ *6 mV/dB*, this causes an error in compression gain of around *2.5dB*, for infinite compression. Thus the *dc error* at the RMS detector output translates into a *compression error* at the compressor output.

This error can be calculated by substituting equations (44) and (60) in (59). Ultimately the *compression error* can be expressed as a VCA gain error, in dB, by applying the well known formula $20\log(g_{error})$, where g_{error} is the gain error. Therefore the compression error in dB is:

$$er_{compression} = \frac{10 \cdot k}{\ln(10)} \sum_n \frac{1}{2n\left(1 + 4\omega^2\tau^2\right)^{2n}} \frac{(2n-1)(2n-3)(2n-5)\cdots 3 \cdot 1}{2n(2n-2)(2n-4)\cdots 4 \cdot 2} \text{ [dB]} \qquad (69)$$

where $n = 1, 2, 3, \ldots$ and k is the same *dc* gain factor as shown in Fig. 8.

Fig. 11 shows a plot of equation (69) for two time constants: normal *35ms* and fast *17μs*. As expected, for the fast time constant, the error is significant.

3 Conclusion

Compressor and expander attack and release time constants must be set according to the application. If a specific sonic effect is desired, mathematical tools are not very useful. However, other applications that require more exact knowledge of the output envelope, such as speaker protection and broadcasting will benefit from exact "tools" to calculate signal transients. For example, speaker manufacturer could specify maximum attack time and tailor it for best sonic performance. In broadcasting, long-term and transient overmodulation are concerns. In these cases, the compressor can be easily simulated using the formulae derived in this paper.

In sections 1.3.3 and 2.3 it is shown the influence of the fast time constant to the precision of the RMS detector and compressor-limiter. Most importantly, the compression factor becomes a function of frequency creating a compression error at low frequencies. This error can be minimized in two ways. One solution is to increase the RMS detector time constant up to a value where the low frequency compression error is acceptable. Another solution is to decrease further more the time constant and extend the *dc error* to high frequencies as well. This is possible because if $\tau \to 0$, in equation (69), then the *dc error* is constant and independent of frequency. Also, note that if the RMS detector and VCA track with temperature, the *dc error* is independent of temperature as well.

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5 References

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Appendix I

Appendix I describes the detail of solving the log filter and the low pass filter differential equations [10]. Let us start with the log filter equation, (11) in the main text:

$$\frac{i_{in}^2(t)}{I_R^2} = \tau\frac{d}{dt}\left(\exp\left(\frac{V_{rms}(t)}{V_T}\right)\right) + \exp\left(\frac{V_{rms}(t)}{V_T}\right) \qquad \text{(AI 1)}$$

Let us define *y(t)* as:

$$y(t) = \exp\left(\frac{V_{rms}(t)}{V_T}\right) \qquad \text{(AI 2)}$$

By substituting equation (AI 2) into differential equation (AI 1), equation (AI 1) becomes:

$$\frac{dy(t)}{dt} + \frac{1}{\tau}y(t) = \frac{1}{\tau}\frac{i_{in}^2(t)}{I_R^2} \qquad \text{(AI 3)}$$

In order to easily integrate the left term of equation (AI 3) it is useful to write it as one term. This can be done by multiplying the left term by a particular exponential function as follows:

$$\left(\frac{dy(t)}{dt} + \frac{1}{\tau}y(t)\right)\exp(u(t)) = \frac{d}{dt}\left(y(t)\cdot\exp(u(t))\right) \qquad \text{(AI 4)}$$

The correct function *u(t)* is found by expanding the right term of (AI 4) and comparing it to the left term:

$$\begin{aligned}&\frac{d}{dt}\left(y(t)\cdot\exp(u(t))\right) = \exp(u(t))\cdot\left(\frac{dy(t)}{dt}\right) + y(t)\cdot\exp(u(t))\frac{du(t)}{dt} = \\ &\left(\left(\frac{dy(t)}{dt}\right) + y(t)\frac{du(t)}{dt}\right)\cdot\exp(u(t))\end{aligned} \qquad \text{(AI 5)}$$

Comparing the last part of (AI 5) to the left term of equation (AI 4), *u(t)* must satisfy:

$$\frac{du(t)}{dt} = \frac{1}{\tau} \qquad \text{(AI 6)}$$

in order to satisfy equation (AI 4).

The solution of equation (AI 6) follows:

$$u(t) = \int\frac{1}{\tau}\cdot dt = \frac{t}{\tau} \qquad \text{(AI 7)}$$

Now multiply both terms of equation (AI 3) by *exp(u(t))*:

$$\left(\frac{dy(t)}{dt} + \frac{1}{\tau}y(t)\right)\exp(u(t)) = \left(\frac{1}{\tau}\frac{i_{in}^2(t)}{I_R^2}\right)\exp(u(t)) \qquad \text{(AI 8)}$$

Substituting the definition of $u(t)$ from (AI 7) in (AI 8) and applying the simplification of (AI 4), equation (AI 8) can be written as:

$$\frac{d}{dt}\left(y(t)\cdot\exp\left(\frac{t}{\tau}\right)\right)=\left(\frac{1}{\tau}\frac{i_{in}^2(t)}{I_R^2}\right)\exp\left(\frac{t}{\tau}\right) \tag{AI 9}$$

Equation (AI 9) can be integrated as follows:

$$y(t)\cdot\exp\left(\frac{t}{\tau}\right)=\int\left(\frac{1}{\tau}\frac{i_{in}^2(t)}{I_R^2}\right)\exp\left(\frac{t}{\tau}\right)\cdot dt+c \tag{AI 10}$$

where c is a constant.

Furthermore (AI 10) can be arranged as follows:

$$y(t)=\left(\int\left(\frac{1}{\tau}\frac{i_{in}^2(t)}{I_R^2}\right)\exp\left(\frac{t}{\tau}\right)\cdot dt+c\right)\cdot\exp\left(-\frac{t}{\tau}\right) \tag{AI 11}$$

By replacing $y(t)$ from (AI 2), (AI 11) becomes:

$$\exp\left(\frac{V_{rms}(t)}{V_T}\right)=\left(\int\frac{1}{\tau}\frac{i_{in}^2(t)}{I_R^2}\exp\left(\frac{t}{\tau}\right)\cdot dt+c\right)\exp\left(-\frac{t}{\tau}\right) \tag{AI 12}$$

Taking the logarithm of the left and right terms, equation (AI 12) can be written as:

$$\frac{V_{rms}(t)}{V_T}=\ln\left(\left(\int\frac{1}{\tau}\frac{i_{in}^2(t)}{I_R^2}\exp\left(\frac{t}{\tau}\right)\cdot dt+c\right)\exp\left(-\frac{t}{\tau}\right)\right) \tag{AI 13}$$

One of the properties of a logarithm is that the logarithm of a product of terms is equal to the sum of the logarithms of each term. Mathematically,

$$\ln(ab)=\ln(a)+\ln(b) \tag{AI 14}$$

So, equation (AI 13) can be arranged as follows:

$$V_{rms}(t)=-\frac{V_T}{\tau}t+2V_T\ln\left(\sqrt{\int\frac{1}{\tau}\frac{i_{in}^2(t)}{I_R^2}\exp\left(\frac{t}{\tau}\right)\cdot dt+c}\right) \tag{AI 15}$$

The differential equation of the low pass filter (linear RMS detector), (4) in the main text, is similar to equation (AI 3). Therefore, it can be solved using the same steps from (AI 3) to (AI 11). Replacing $\frac{i_{in}^2(t)}{I_R^2}$ by $v_{in}^2(t)$ and $y(t)$ by $V_2(t)$ in (AI 3), the solution to differential equation (4) follows:

$$V_2(t)=\left(\int\frac{1}{\tau}v_{in}^2(t)\cdot\exp\left(\frac{t}{\tau}\right)\cdot dt+c\right)\cdot\exp\left(-\frac{t}{\tau}\right) \tag{AI 16}$$

Let us assume that the input voltage, $v_{in}^2(t)$, is:

$$v_{in}(t)=V_0\cos(\omega t) \tag{AI 17}$$

Substituting (AI 17) in equation (AI 16) $V_2(t)$ is:

$$V_2(t) = \left(\frac{V_0^2}{\tau}\int \cos^2(\omega t)\cdot \exp\left(\frac{t}{\tau}\right)\cdot dt + c\right)\cdot \exp\left(-\frac{t}{\tau}\right) = \left(\frac{V_0^2}{2\tau}\int (1+\cos(2\omega t))\cdot \exp\left(\frac{t}{\tau}\right)\cdot dt + c\right)\cdot \exp\left(-\frac{t}{\tau}\right) =$$

$$= \left(\frac{V_0^2}{2\tau}\left(\int \exp\left(\frac{t}{\tau}\right)\cdot dt + \int \cos(2\omega t)\cdot \exp\left(\frac{t}{\tau}\right)\cdot dt\right) + c\right)\cdot \exp\left(-\frac{t}{\tau}\right)$$

(AI 18)

The above integral can be calculated separately. Let's define E as:

$$E = E_1 + E_2 \qquad \text{(AI 19)}$$

where:

$$E_1 = \int \exp\left(\frac{t}{\tau}\right)\cdot dt \qquad \text{(AI 20)} \quad ; \quad E_2 = \int \cos(2\omega t)\cdot \exp\left(\frac{t}{\tau}\right)\cdot dt \qquad \text{(AI 21)}$$

The solution to equation (AI 20) is:

$$E_1 = \int \exp\left(\frac{t}{\tau}\right)\cdot dt = \tau\cdot \exp\left(\frac{t}{\tau}\right) \qquad \text{(AI 22)}$$

Equation (AI 21) can be solved as follows:

$$E_2 = \int \cos(2\omega t)\cdot \exp\left(\frac{t}{\tau}\right)\cdot dt = \tau\cdot \exp\left(\frac{t}{\tau}\right)\cos(2\omega t) - \tau\int \exp\left(\frac{t}{\tau}\right)(-2\omega)\sin(2\omega t)\cdot dt =$$

$$= \tau\cdot \exp\left(\frac{t}{\tau}\right)\cos(2\omega t) + 2\omega\tau\left(\tau\cdot \exp\left(\frac{t}{\tau}\right)\sin(2\omega t) - 2\omega\tau\int \cos(2\omega t)\cdot \exp\left(\frac{t}{\tau}\right)\cdot dt\right)$$

(AI 23)

Notice that the last integral of (AI 23) is exactly E_2. Thus, (AI 23) can be written as:

$$E_2 = \tau\cdot \cos(2\omega t)\cdot \exp\left(\frac{t}{\tau}\right) + 2\omega\tau^2 \sin(2\omega t)\cdot \exp\left(\frac{t}{\tau}\right) - 4\omega^2\tau^2\cdot E_2 \qquad \text{(AI 24)}$$

Equation (AI 24) can be solved for E_2 as follows:

$$E_2 = \tau\cdot \exp\left(\frac{t}{\tau}\right)\cdot \frac{\cos(2\omega t) + 2\omega\tau\cdot \sin(2\omega t)}{1+4\omega^2\tau^2} \qquad \text{(AI 25)}$$

Substituting equation (AI 22) for E_1 and equation (AI 25) for E_2, E becomes:

$$E = \tau\cdot \exp\left(\frac{t}{\tau}\right)\cdot \left(1 + \frac{\cos(2\omega t) + 2\omega\tau\cdot \sin(2\omega t)}{1+4\omega^2\tau^2}\right) \qquad \text{(AI 26)}$$

The sum of integrals in equation (AI 18) can now be substituted by its solution, equation (AI 26), as follows:

$$V_2(t) = \left(\frac{V_0^2}{2\tau}\cdot\tau\cdot\exp\left(\frac{t}{\tau}\right)\cdot\left(1+\frac{\cos(2\omega t)+2\omega\tau\cdot\sin(2\omega t)}{1+4\omega^2\tau^2}\right)+c\right)\cdot\exp\left(-\frac{t}{\tau}\right) =$$
$$= \frac{V_0^2}{2}\cdot\left(1+\frac{\cos(2\omega t)+2\omega\tau\cdot\sin(2\omega t)}{1+4\omega^2\tau^2}\right)+c\cdot\exp\left(-\frac{t}{\tau}\right) \tag{AI 27}$$

Let us define the following function:

$$G(\omega,t) = 1+\frac{\cos(2\omega t)+2\omega\tau\cdot\sin(2\omega t)}{1+4\omega^2\tau^2} \tag{AI 28}$$

Substituting (AI 28) in (AI 27) the following equation is obtained:

$$V_2(t) = \frac{V_0^2}{2}\cdot G(\omega,t)+c\cdot\exp\left(-\frac{t}{\tau}\right) \tag{AI 29}$$

Also, notice that for $\omega\tau >> 1$, equation (AI 28) can be approximated to:

$$G(\omega,t) = 1+\frac{\sin(2\omega t)}{2\omega\tau} \tag{AI 30}$$

Appendix II

The reduction formula for equation (37) in the main text is found by partial integration.

$$
\begin{aligned}
E &= \int \cos^n(y)\cdot dy = \int \cos(y)\cos^{n-1}(y)\cdot dy = \\
&\sin(y)\cos^{n-1}(y) - (n-1)\int \sin(y)\cos^{n-2}(y)(-\sin(y))\cdot dy = \\
&\sin(y)\cos^{n-1}(y) + (n-1)\int \left(1-\cos^2(y)\right)\cos^{n-2}(y)\cdot dy = \\
&\sin(y)\cos^{n-1}(y) + (n-1)\int \cos^{n-2}(y)\cdot dy - (n-1)\int \cos^n(y)\cdot dy
\end{aligned}
\qquad \text{(AII 1)}
$$

Last term of above equation is the same as the expression E scaled by *(n-1)*. Thus:

$$
E = \int \cos^n(y)\cdot dy = \frac{\sin(y)}{n}\cos^{n-1}(y) + \frac{n-1}{n}\int \cos^{n-2}(y)\cdot dy \qquad \text{(AII 2)}
$$

We can again apply the reduction formula (AII 2) to the last term of above equation:

$$
E = \int \cos^n(y)\cdot dy = \frac{\sin(y)}{n}\cos^{n-1}(y) + \frac{n-1}{n}\left(\frac{\sin(y)}{n-2}\cos^{n-3}(y) + \frac{n-3}{n-2}\int \cos^{n-4}(y)\cdot dy\right) =
$$

$$
\frac{1}{n}\sin(y)\cos^{n-1}(y) + \frac{n-1}{n(n-2)}\sin(y)\cos^{n-3}(y) + \frac{(n-1)(n-3)}{n(n-2)}\int \cos^{n-4}(y)\cdot dy
$$

(AII 3)

If n is odd, the last term of (AII 3) will always be the integral of *cos(y)*. The closed form solution of the last term is $\int cos(y)\cdot dy = sin(y)$. Note that all terms of formula (AII 3) are multiplied by *sin(y)*. Therefore for n odd, E can be written as:

$$
E_{odd} = \sin(y)\left(\frac{\cos^{n-1}(y)}{n} + \frac{n-1}{n(n-2)}\cos^{n-3}(y) + \frac{(n-1)(n-3)}{n(n-2)(n-4)}\cos^{n-5}(y) + \cdots + \frac{(n-1)(n-3)\cdots 4\cdot 2}{n(n-2)(n-4)\cdots 3\cdot 1}\right)
$$

(AII 4)

E_{odd} has $\frac{n+1}{2}$ terms.

If n is even, the last term of (AII 3) will always be the integral of *dy*. The closed form solution of the last term is $\int dy = y$. Note that all terms of formula (AII 3), but the last one, are multiplied by *sin(y)*. Therefore for n even, E can be written as:

$$
E_{even} = \sin(y)\left(\frac{\cos^{n-1}(y)}{n} + \frac{n-1}{n(n-2)}\cos^{n-3}(y) + \cdots + \frac{(n-1)(n-3)\cdots 5\cdot 3}{n(n-2)(n-4)\cdots 4\cdot 2}\cos(y)\right) + \frac{(n-1)(n-3)\cdots 5\cdot 3\cdot 1}{n(n-2)(n-4)\cdots 4\cdot 2}
$$

(AII 5)

E_{even} has $\frac{n}{2}+1$ terms.

E_{odd} and E_{even} have *cosine* terms multiplied by *sin(y)*. But the product of $sin(y)cos^k(y)$ can be written as a sum of higher harmonics of *sin(y)*. Notice that:

$$\sin(y)\cos^k(y)=\frac{1}{2}\sin(y)(1+\cos(2y))\cos^{k-2}(y)=\left(\frac{1}{2}\sin(y)+\frac{1}{2}\sin(y)\cos(2y)\right)\cos^{k-2}(y)=$$

$$=\left(\frac{1}{2}\sin(y)+\frac{1}{4}(\sin(3y)-\sin(y))\right)\cos^{k-2}(y)=\frac{1}{4}\sin(y)\cos^{k-2}(y)+\frac{1}{4}\sin(3y)\cos^{k-2}(y)=$$

$$=\frac{1}{4}\left(\frac{1}{4}\sin(y)\cos^{k-4}(y)+\frac{1}{4}\sin(3y)\cos^{k-4}(y)\right)+\frac{1}{4}\left(\frac{1}{2}\sin(3y)(1+\cos(2y))\cos^{k-4}(y)\right)=$$

$$=\frac{1}{4}\left(\frac{1}{4}\sin(y)\cos^{k-4}(y)+\frac{1}{4}\sin(3y)\cos^{k-4}(y)\right)+\frac{1}{4}\left(\frac{1}{4}\sin(y)+\frac{1}{2}\sin(3y)+\frac{1}{4}\sin(5y)\right)\cos^{k-4}(y)=$$

$$=\left(\frac{1}{8}\sin(y)+\frac{3}{16}\sin(3y)+\frac{1}{16}\sin(5y)\right)\cos^{k-4}(y) \qquad \text{(AII 6)}$$

where the following trigonometric property is used:

$$\sin(a)\cos(b)=\frac{1}{2}(\sin(a+b)+\sin(a-b)) \qquad \text{(AII 7)}$$

and $k = 1,2,3...$.

The same reduction method can be applied one more time:

$$\sin(y)\cos^k(y)=\left(\frac{5}{64}\sin(y)+\frac{9}{64}\sin(3y)+\frac{5}{64}\sin(5y)+\frac{1}{64}\sin(7y)\right)\cos^{k-6}(y) \qquad \text{(AII 8)}$$

It can be seen that $sin(y)cos^k(y)$ is reduced to a sum of odd harmonics of *sin(y)* multiplied by the factor $cos^m(y)$, where m is an arbitrary positive integer. Equation (AII 7) can be written differently if k is odd or even.

For even k, the cosine term becomes *one* (because it is raised to the power *zero*) and equation (AII 8) can be written as:

$$\sin(y)\cos^k(y)=\sum_{i=1}^{\frac{k}{2}+1}c_i\sin((2i-1)y) \qquad \text{(AII 9)}$$

where c_i are real coefficients.

For odd k, the cosine term becomes *cos(y)* (because it is raised to the power *one*) and equation (AII 8) can be written as:

$$\sin(y)\cos^k(y)=\cos(y)\left(\sum_{i=1}^{\frac{k+1}{2}}c_i\sin((2i-1)y)\right) \qquad \text{(AII 10)}$$

where c_i are real coefficients.

Taking into account trigonometric property (AII 7), equation (AII 10) can be written as:

$$\sin(y)\cos^k(y) = \sum_{i=1}^{\frac{k+1}{2}} \frac{c_i}{2}\left(\sin(2iy) + \sin((2i-2)y)\right) \qquad \text{(AII 11)}$$

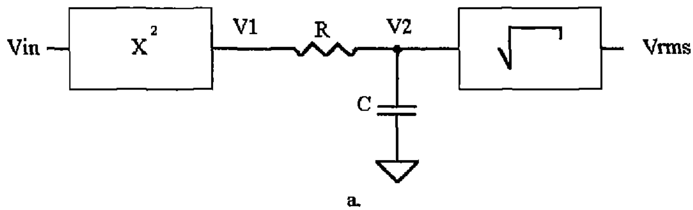
Therefore, *sin(y)cos^k(y)* can be described as a sum of higher harmonics of *sin(y)*, for any positive and integer *k*, as follows:

$$\sin(y)\cos^k(y) = \sum_i c_i \sin(i \cdot y) \qquad \text{(AII 12)}$$

Then the integral of *sin(y)cos^k(y)* over the one signal period is:

$$\int_0^{2\pi} \sin(y)\cos^k(y) \cdot dy = \int_0^{2\pi} \sum_i c_i \sin(i \cdot y) \cdot dy = \sum_i c_i \int_0^{2\pi} \sin(i \cdot y) \cdot dy = 0 \qquad \text{(AII 13)}$$

Thus, the integral of E_{odd} (AII 4) over the signal period of time is *zero*.



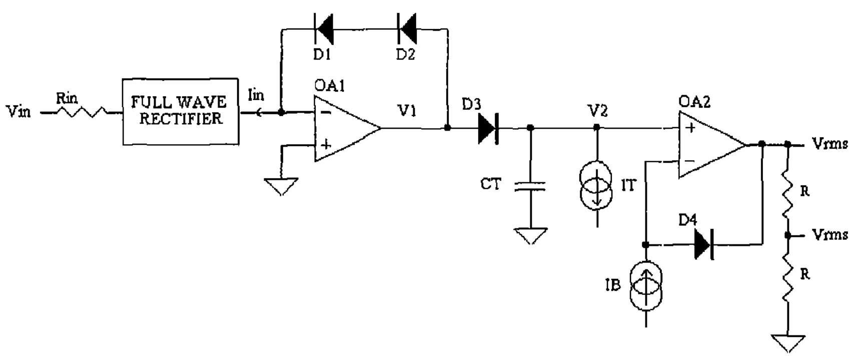
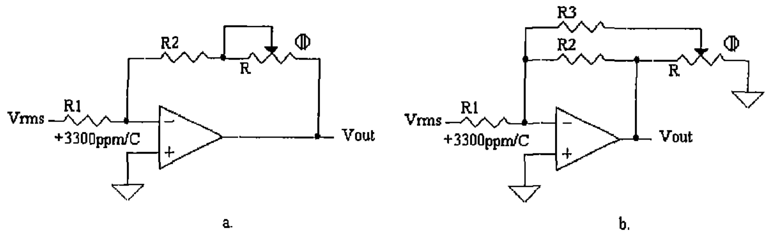


**Figure 1. a) RMS detector blocks with filter in the linear domain.
b) Simplified circuit diagram of the RMS detector with filter in the log domain.**



$$\text{gain} = -\frac{R2 + R(1+\Delta T \cdot \text{tempcopot})}{R1(1+\Delta T \cdot 0.0033)}$$

$$\text{gain} \cong -\frac{\left(R2 \| \frac{R3}{(1-\alpha)}\right)}{R1(1+\Delta T \cdot 0.0033)}$$

where α is potentiometer position.

Figure 2. RMS detector external temperature compensation.

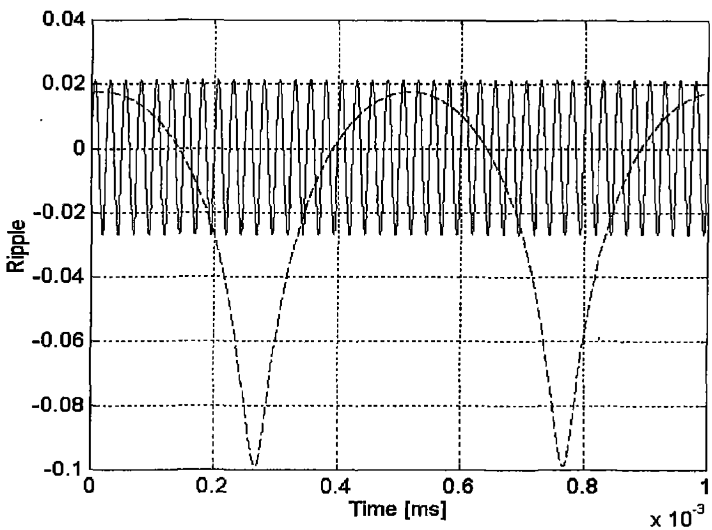


Figure 3. RMS detector output ripple, solid ***f = 20kHz,*** **dashed** ***f = 1kHz, τ = 17μs.***

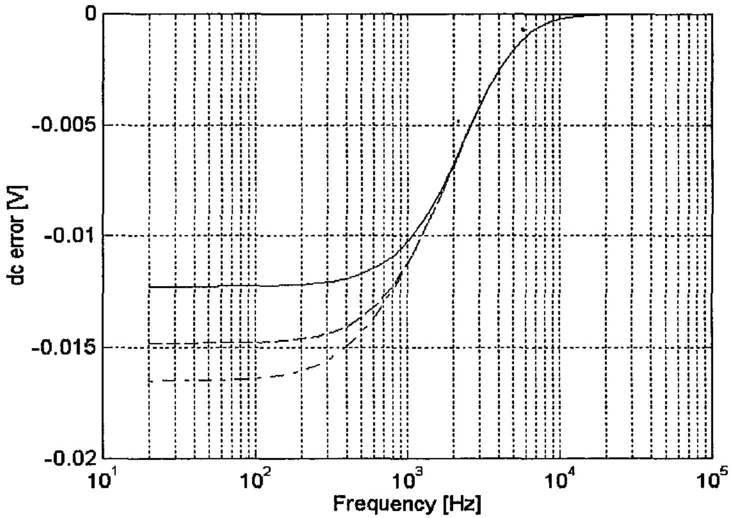


Figure 4. ***dc error*** **calculated using equation (44), solid 5 terms, dashed 20 terms, dash dot 100 terms.**

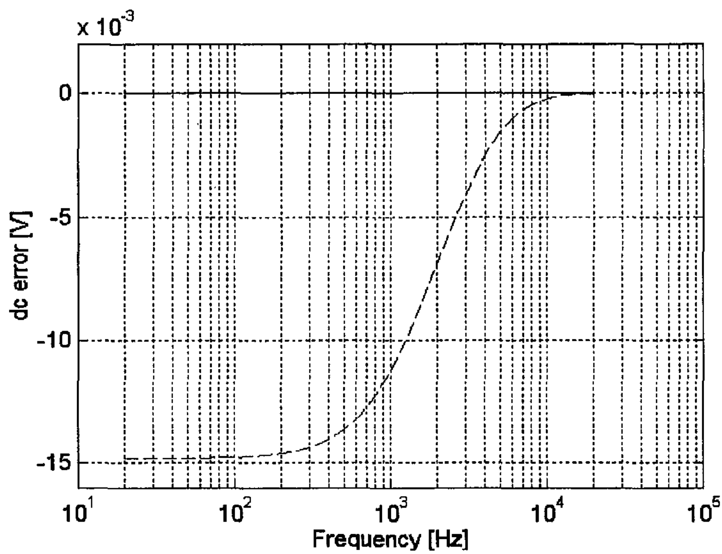


Figure 5. ***dc error*, solid $\tau = 35ms$, dashed $\tau = 17\mu s$, 20 terms.**

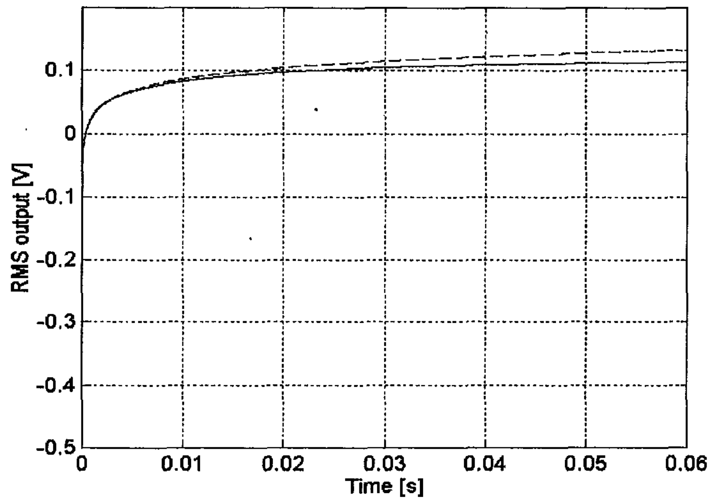


Figure 6. a) RMS detector *dc* output voltage during attack time, solid equation (49), dashed approximation equation (51) for $t \ll \tau$, $\tau = 35ms$, $V_P = -0.4V$, input level *20dB* above reference level.

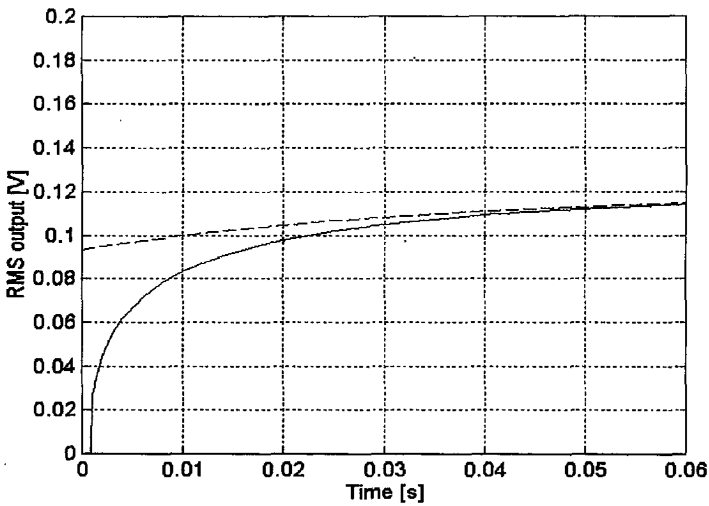


Figure 6. b) solid equation (49), dashed approximation equation (52) for $t \gg \tau$.

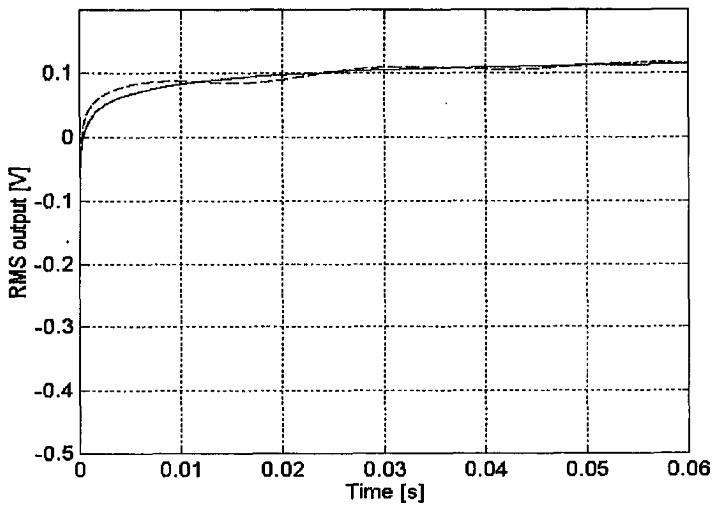


Figure 6. c) solid equation (49), dashed general equation (48) for $f = 20Hz$, $\tau = 35ms$.

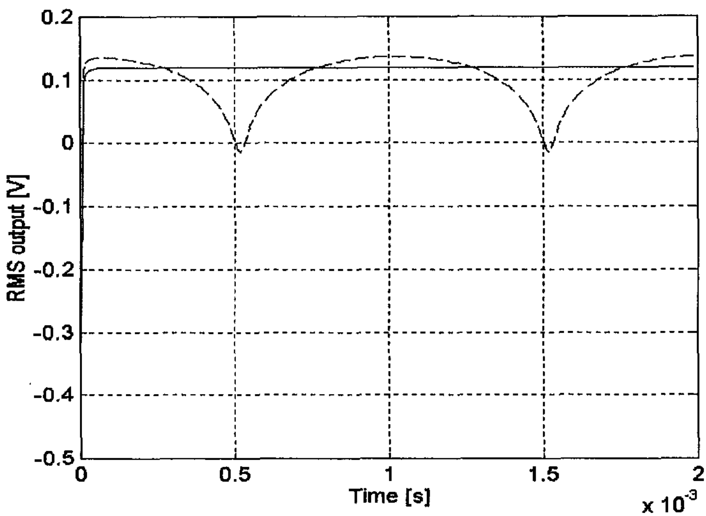


Figure 6. d) solid equation (49), dashed general equation (48) for $f = 500Hz$, $\tau = 17\mu s$.

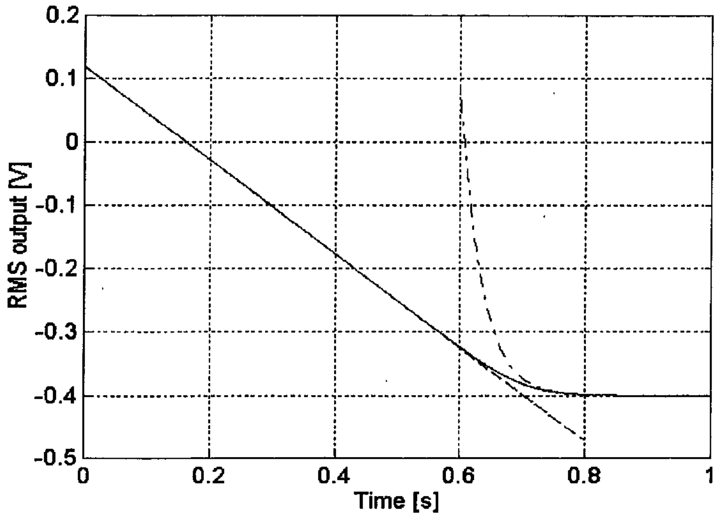
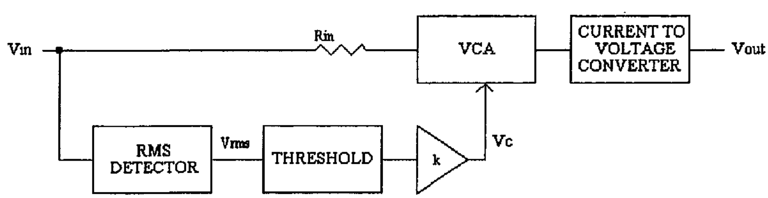
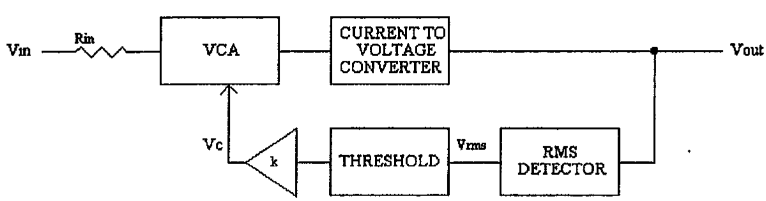


Figure 7. RMS detector *dc* output voltage during release time, solid equation (56), dashed equation (57), dash dot equation (58).



a.



b.

Figure 8. a) Feedforward compressor. b) Feedback compressor.

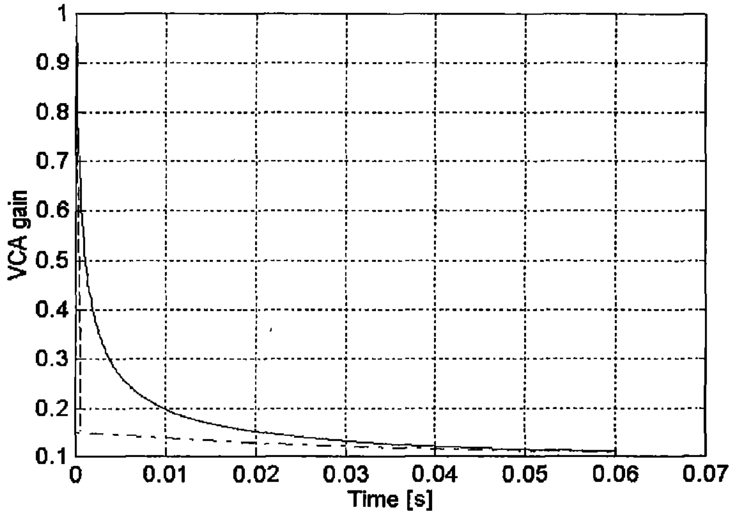


Figure 9. a) VCA output envelope during attack time, dashed for *t* → *0*, dash dot approximation for *t* >> *τ*, *τ* = *35ms*, *20dB* final gain reduction.

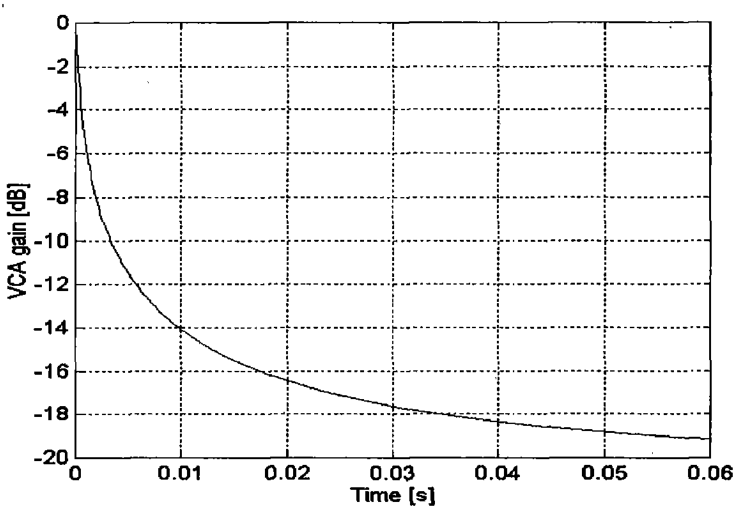


Figure 9. b) VCA output envelope during attack time in decibels.

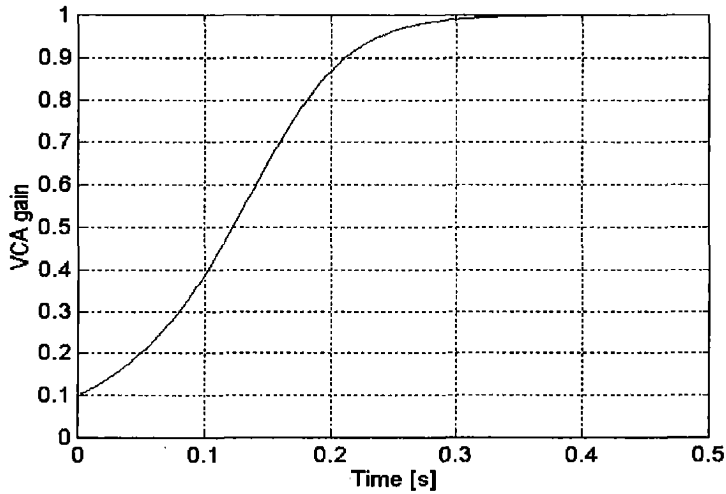


Figure 10. a) VCA output envelope during release time, $\tau = 35ms$, *20dB* initial gain reduction.

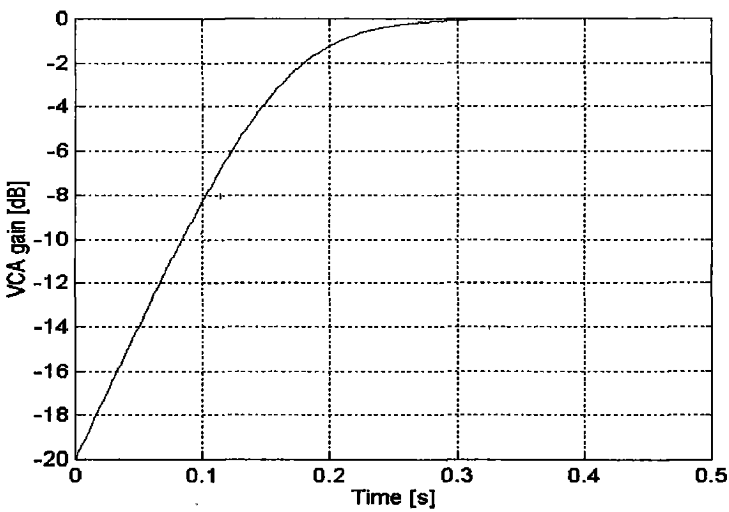


Figure 10. b) VCA output envelope during release time in decibels.

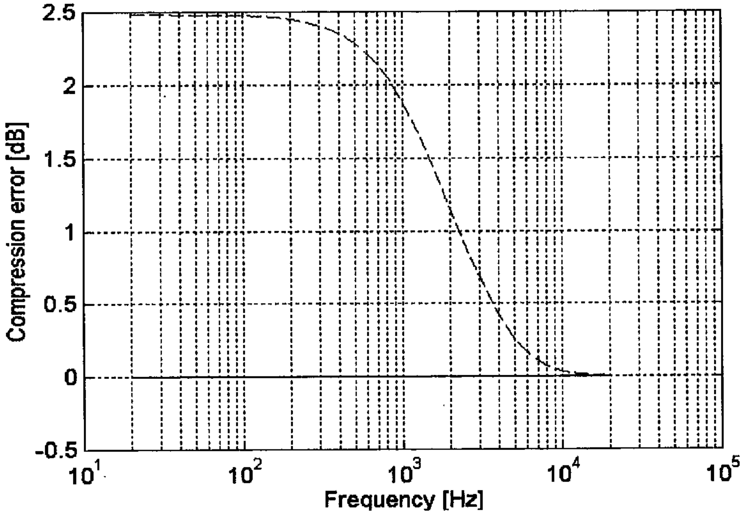


Figure 11. Compression error due to different time constants, solid $\tau = 35ms$, dashed $\tau = 17\mu s$.